Maximum Entropy

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Roadmap

- Why we need more powerful probabilistic modeling formalism
- Introducing key concepts from information theory
- Maximum Entropy Models
  - Why they have the form they do
  - How to estimate them from data
Outline

Motivation: Document Classification

Expectation and Entropy

Constraints

Maximum Entropy Form
Modeling Distributions

- Modeling Distributions
- Estimating from data
- Thus far, only counting
  - MLE
  - Priors
  - Backoff
Motivation: Document Classification

Modeling Distributions

• Modeling Distributions
• Estimating from data
• Thus far, only counting
  ○ MLE
  ○ Priors
  ○ Backoff
• What about features?
Supervised Learning

- **Problem Setup**
  - Given: some annotated data
  - Goal: Build a model
  - Task: Apply it to unseen data

- **Issues**
  - More data help
  - How to represent the data
Supervised Learning

- **Problem Setup**
  - Given: some annotated data
  - Goal: Build a model
  - Task: Apply it to unseen data

- **Issues**
  - More data help
  - How to represent the data

- **Better document labeling**
Supervised Learning

- **Problem Setup**
  - Given: some annotated data (document categories)
  - Goal: Build a model (using some feature representation)
  - Task: Apply it to unseen data (document labeling)

- **Issues**
  - More data help
  - How to represent the data

- **Better document labeling**
Motivation: Document Classification

Contrast: Naive Bayes

- **NB useful and simple; two parameters**
  - Prior distribution
  - Conditional emission
- **Training is easy from tagged data (counting)**
- **Find best label using arithmetic**
Contrast: Naive Bayes

- NB useful and simple; two parameters
  - Prior distribution
  - Conditional emission
- Training is easy from tagged data (counting)
- Find best label using arithmetic
- But it ignores important clues that could help
Motivating Example: Document Classification

- We can do better than counting words
Motivating Example: Document Classification

- We can do better than counting words
- Bigrams
- Metadata
- Morphology (Apple documents: how many words start with i)
- Style (length of sentences)
- Number of repeated sentences
Motivation: Document Classification

Encoding Features

- Much more powerful and expressive than counting single observations
  - $\vec{f}(x)$ a vector with the feature count for observation $x$
  - $f_i(x)$: count of feature $i$ in observation $x$
Motivation: Document Classification

Encoding Features

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- Typical example

\[ f(w_1, w_2, w_3, w_4, \ldots w_n) = \]
Motivation: Document Classification

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- Typical example

\[
f(w_1, w_2, w_3, w_4, \ldots w_n) =
\]

- $f_1(\vec{w})$ Number of times you see the word “dog”
- $f_2(\vec{w})$ Number of times you see the bigram “dog house”
Motivation: Document Classification

Where Maximum Entropy Models Fit

• Suppose we have some data-driven information about these features
• What distribution should we use to model these features?
• “Maximum Entropy” models provide a solution
Where Maximum Entropy Models Fit

• Suppose we have some data-driven information about these features
• What distribution should we use to model these features?
• “Maximum Entropy” models provide a solution . . . but first quick refresher
Outline

Motivation: Document Classification

Expectation and Entropy

Constraints

Maximum Entropy Form
Entropy & Expectation

An *expectation* of a random variable is a weighted average:

\[
\mathbb{E} [f(X)] = \sum_{x=1}^{\infty} f(x) p(x)
\]

(*discrete*)

*Entropy* is a measure of uncertainty that is associated with the distribution of a random variable:

\[
H(X) = -\mathbb{E} [\lg(p(X))]
\]

\[
= - \sum_{x} p(x) \lg(p(x))
\]

(*discrete*)
Principles for Modeling Distributions

Maximum Entropy Principle (Jaynes)

All else being equal, we should prefer distributions that maximize the Entropy
Principles for Modeling Distributions

Maximum Entropy Principle (Jaynes)

All else being equal, we should prefer distributions that maximize the Entropy

- What additional constraints do we want to place on the distribution?
- How, mathematically, do we optimize the entropy?
Outline

Motivation: Document Classification

Expectation and Entropy

Constraints

Maximum Entropy Form
The obvious one . . .

- We’re attempting to model a probability distribution \( p \).
- By definition, our probability distribution must sum to one

\[
\sum_x p(x) = 1 \quad (1)
\]
Feature constraints

- We observe features across many outcomes
- We’re modeling a distribution $p$ over observations $x$. What is the correct model of features under this distribution?
- The whole point of this is that we don’t want to count outcomes (we’ve discussed those methods)
Feature constraints

- We observe features across many outcomes
- We’re modeling a distribution $p$ over observations $x$. What is the correct model of features under this distribution?
- The whole point of this is that we don’t want to count outcomes (we’ve discussed those methods)
- Ideally, the expected count of the features should be consistent with observations

### Estimated Counts

$$ \mathbb{E}_p [f_i(x)] = \sum_x p(x)f_i(x) \quad (2) $$

### Empirical Counts

$$ \mathbb{E}_{\hat{p}} [f_i(x)] = \hat{p}(x)f_i(x) \quad (3) $$

- Empirical distribution is just what we’ve observed in data
Optimizing Constrained Functions

Theorem: Lagrange Multiplier Method

Given functions \( f(x_1, \ldots, x_n) \) and \( g(x_1, \ldots, x_n) \), the critical points of \( f \) restricted to the set \( g = 0 \) are solutions to equations:

\[
\frac{\partial f}{\partial x_i}(x_1, \ldots, x_n) = \lambda \frac{\partial g}{\partial x_i}(x_1, \ldots, x_n) \quad \forall i
\]

\[
g(x_1, \ldots, x_n) = 0
\]

This is \( n + 1 \) equations in the \( n + 1 \) variables \( x_1, \ldots x_n, \lambda \).
Lagrange Example

Maximize \( f(x, y) = \sqrt{xy} \) subject to the constraint \( 20x + 10y = 200 \).

- Compute derivatives
Lagrange Example

Maximize $f(x, y) = \sqrt{xy}$ subject to the constraint $20x + 10y = 200$.

- Compute derivatives

$$\frac{\partial f}{\partial x} = \frac{1}{2} \sqrt{\frac{y}{x}} \quad \frac{\partial g}{\partial x} = 20$$

$$\frac{\partial f}{\partial y} = \frac{1}{2} \sqrt{\frac{x}{y}} \quad \frac{\partial g}{\partial y} = 10$$
Lagrange Example

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- Create new systems of equations
Lagrange Example

Maximize \( f(x, y) = \sqrt{xy} \) subject to the constraint \( 20x + 10y = 200 \).

- Compute derivatives
  
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  \frac{\partial f}{\partial x} = \frac{1}{2} \sqrt{\frac{y}{x}} \quad \frac{\partial g}{\partial x} = 20
  \]
  
  \[
  \frac{\partial f}{\partial y} = \frac{1}{2} \sqrt{\frac{x}{y}} \quad \frac{\partial g}{\partial y} = 10
  \]

- Create new systems of equations

  \[
  \frac{1}{2} \sqrt{\frac{y}{x}} = 20\lambda
  \]
  
  \[
  \frac{1}{2} \sqrt{\frac{x}{y}} = 10\lambda
  \]

  \[
  20x + 10y = 200
  \]
Lagrange Example

- Dividing the first equation by the second gives us

\[
\frac{y}{x} = 2
\]  \hspace{1cm} (4)

- which means \( y = 2x \), plugging this into the constraint equation gives:

\[
20x + 20(2x) = 200
\]

\[
x = 5 \Rightarrow y = 10
\]
Outline

Motivation: Document Classification

Expectation and Entropy

Constraints

Maximum Entropy Form
Objective Function

- We want a distribution $p$ that maximizes

$$H(p) \equiv - \sum_x p(x) \log p(x)$$  \hfill (5)

- Under the constraints that

$$\sum_x p(x) = 1$$  \hfill (6)

- and, for every feature $f_i$

$$\mathbb{E}_p [f_i] = \hat{\mathbb{E}}_{\hat{p}} [f_i].$$  \hfill (7)
Augmented Objective Function

\[ \Lambda(p, \lambda, \gamma) = \]
\[ - \sum_x p(x) \log p(x) \]
\[ - \sum_i \lambda_i \left( \sum_x p(x)f_i(x) - \hat{E}[f_i] \right) \]
\[ - \gamma \left( \sum_x p(x) - 1 \right) \]

Plan for solution:
- Take derivative
- Set it equal to zero
- Solve for the \( p(x) \) that optimizes equation
- This will give the functional form of our solution
Solution

\[ 0 = \frac{\partial \Lambda(p, \lambda, \gamma)}{\partial p(x)} \]  \hspace{1cm} (8)

\[ 0 = -(1 + \log(p(x))) + \sum_i \lambda_i f_i(x) + \gamma \]  \hspace{1cm} (9)

\[ \log(p(x)) = \sum_i \lambda_i f_i(x) + \gamma - 1 \]  \hspace{1cm} (10)
Solution

\[ 0 = \frac{\partial \Lambda(p, \lambda, \gamma)}{\partial p(x)} \quad (8) \]

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Now solve for \( p(x) \):

\[ p(x) = \exp\{\gamma - 1\} \exp\left\{ \sum_i \lambda_i f_i(x) \right\} \quad (11) \]
Solution

Substitute into normalization (Equation 6):

$$\sum_x p(x) = 1$$  \hspace{1cm} (12)
Solution

Substitute into normalization (Equation 6):

\[
\sum_x p(x) = 1 \quad (12)
\]

\[
\sum_x \exp \left\{ \gamma - 1 \right\} \exp \left\{ \sum_i \lambda_i f_i(x) \right\} = 1 \quad (13)
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Solution

Substitute into normalization (Equation 6):

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\sum_x \exp \{\gamma - 1\} \exp \left\{ \sum_i \lambda_i f_i(x) \right\} = 1 \quad (13)
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\[
\exp \{\gamma - 1\} \sum_x \exp \left\{ \sum_i \lambda_i f_i(x) \right\} = 1 \quad (14)
\]
Solution

Substitute into normalization (Equation 6):

\[ \sum_x p(x) = 1 \quad (12) \]

\[ \sum_x \exp \{ \gamma - 1 \} \exp \left\{ \sum_i \lambda_i f_i(x) \right\} = 1 \quad (13) \]

\[ \exp \{ \gamma - 1 \} \sum_x \exp \left\{ \sum_i \lambda_i f_i(x) \right\} = 1 \quad (14) \]

\[ \exp \{ \gamma - 1 \} = \frac{1}{\sum_x \exp \left\{ \sum_i \lambda_i f_i(x) \right\}} \quad (15) \]
Final steps!

Substitute Equation 12 into Equation 11:

\[ p(x) = \exp\{\gamma - 1\} \exp\left\{ \sum_i \lambda_i f_i(x) \right\} \]  \hspace{1cm} (16)
Final steps!

Substitute Equation 12 into Equation 11:

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(16)
Final steps!

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Final steps!

Substitute Equation 12 into Equation 11:

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(16)

\[ p(x) = \frac{1}{\sum_x \exp \left\{ \sum_i \lambda_i f_i(x) \right\}} \exp \left\{ \sum_i \lambda_i f_i(x) \right\} \]  

(17)

More concretely:

\[ p(x) = \frac{\exp \left\{ \lambda^\top \vec{f}(x) \right\}}{\sum_{x'} \exp \left\{ \lambda^\top \vec{f}(x) \right\}} \]  

(18)
Form of Solution

• Other ways to arrive at same answer (monkeys throwing balls)
• Should remind you of logistic regression

\[
p(x) = \frac{\exp \left\{ \lambda^\top \bar{f}(x) \right\}}{\sum_{x'} \exp \left\{ \lambda^\top \bar{f}(x') \right\}}
\]

(19)

• Thus, distribution is parameterized by \( \bar{\lambda} \) (one for each feature, used \( \beta \) before)
Finding Parameters

- Form is simple
- However, finding parameter is difficult
- Solutions take iterative form
  1. Start with $\vec{\lambda}^{(0)} = \vec{0}$
  2. For $k = 1 \ldots$
     2.1 Determine update $\vec{\delta}^{(k)}$
     2.2 $\vec{\lambda}^{(k)} \rightarrow \vec{\lambda}^{(k-1)} + \vec{\delta}^{(k)}$
Method for finding updates

- Our objective is a function of $\tilde{\lambda}$

$$L(\lambda) = \sum_x \frac{\exp \{\lambda^T f(x)\}}{\sum_{x'} \exp \{\lambda^T f(x')\}}$$

(20)

(in practice, we typically use the log probability)

- Strategy: Move $\tilde{\lambda}$ by walking up the gradient $G(\lambda^{(k)})$

- Gradient

$$G_i(\lambda) = \frac{\partial L(\lambda)}{\partial \lambda_i} = - \left[ \left( \sum_x p_\lambda(x) f_i(x) \right) - \hat{E} [f_i] \right]$$

(21)
Method for finding updates

- Set the update of the form
  \[ \delta^{(k)} = \alpha^{(k)} G(\lambda^{(k)}) \]  \hspace{1cm} (22)

- Use the new parameter
  \[ \lambda^{(k)} \rightarrow \lambda^{(k-1)} + \delta^{(k)} \]  \hspace{1cm} (23)

- What value of \( \alpha \)?

Maximum Entropy Form
Method for finding updates

- Set the update of the form

\[ \delta^{(k)} = \alpha^{(k)} G(\lambda^{(k)}) \] (22)

- Use the new parameter

\[ \tilde{\lambda}^{(k)} \rightarrow \tilde{\lambda}^{(k-1)} + \delta^{(k)} \] (23)

- What value of \( \alpha \)?
  - Try lots of different values, pick the one that optimizes \( L(\lambda) \) (grid search)
Other parameter estimation techniques

- Iterative scaling
- Conjugate gradient methods
- Real difference is speed and scalability
We often want to prefer small parameters over large ones, all else being equal

\[ L(\lambda) = \sum_x \frac{\exp\left\{ \lambda^\top f(x) \right\}}{\sum_{x'} \exp\left\{ \lambda^\top f(x') \right\}} - \sum_i \frac{\lambda_i^2}{\sigma_i^2} \]  

This is equivalent to having a Gaussian prior on the weights \( \lambda \)

Also possible to use **informed** priors when you have an idea of what the weights should be (e.g. for domain adaptation)
All sorts of distributions

- We talked about a simple distribution \(p(x)\)
- But could just as easily be joint distribution \(p(y, x)\)

\[
p(y, x) = \frac{\exp \left\{ \lambda^\top f(y, x) \right\}}{\sum_{y', x'} \exp \left\{ \lambda^\top f(y', x') \right\}} \tag{25}
\]

- Or a conditional distribution \(p(y|x)\)

\[
p(y|x) = \frac{\exp \left\{ \lambda^\top f(y, x) \right\}}{\sum_{y'} \exp \left\{ \lambda^\top f(y', x) \right\}} \tag{26}
\]
Uses of MaxEnt Distributions

- POS Tagging (state of the art)
- Supervised classification: spam vs. not spam
- Parsing (head or not)
- Many other NLP applications
In class . . .
In class . . .
In class . . .
In class . . .
In class . . .