Probability Distributions: Discrete

Introduction to Data Science Algorithms
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Binomial distribution

- Bernoulli: distribution over two values (success or failure) from a single event
- Binomial: number of successes from multiple Bernoulli events
- Examples:
  - The number of times “heads” comes up after flipping a coin 10 times
  - The number of defective TVs in a line of 10,000 TVs
- Important: each Bernoulli event is assumed to be independent
- Notation: let $X$ be a random variable that describes the number of successes out of $N$ trials.
  - The possible values of $X$ are integers from 0 to $N$: \{0, 1, 2, \ldots, N\}
Binomial distribution

• Suppose we flip a coin 3 times. There are 8 possible outcomes:

\[
P(HHH) = P(H)P(H)P(H) = 0.125 \\
P(HHT) = P(H)P(H)P(T) = 0.125 \\
P(HTH) = P(H)P(T)P(H) = 0.125 \\
P(HTT) = P(H)P(T)P(T) = 0.125 \\
P(THH) = P(T)P(H)P(H) = 0.125 \\
P(THT) = P(T)P(H)P(T) = 0.125 \\
P(TTH) = P(T)P(T)P(H) = 0.125 \\
P(TTT) = P(T)P(T)P(T) = 0.125
\]

• What is the probability of landing heads \(x\) times during these 3 flips?
Binomial distribution

- What is the probability of landing heads \( x \) times during these 3 flips?
  - 0 times:
    - \( P(TTT) = 0.125 \)
  - 1 time:
    - \( P(HTT) + P(THT) + P(TTH) = 0.375 \)
  - 2 times:
    - \( P(HHT) + P(HTH) + P(THH) = 0.375 \)
  - 3 times:
    - \( P(HHH) = 0.125 \)
Binomial distribution

• The probability mass function for the binomial distribution is:

\[ f(x) = \binom{N}{x} \theta^x (1 - \theta)^{N-x} \]

“N choose x”

• Like the Bernoulli, the binomial parameter \( \theta \) is the probability of success from one event.
• Binomial has second parameter \( N \): number of trials.
• The PMF important: difficult to figure out the entire distribution by hand.
Aside: Binomial coefficients

- The expression \( \binom{n}{k} \) is called a \textit{binomial coefficient}.
  - Also called a \textit{combination} in combinatorics.
- \( \binom{n}{k} \) is the number of ways to choose \( k \) elements from a set of \( n \) elements.
- For example, the number of ways to choose 2 heads from 3 coin flips: HHT, HTH, THH
  \( \binom{3}{2} = 3 \)
- Formula:
  \[
  \binom{n}{k} = \frac{n!}{k!(n-k)!}
  \]

Pascal's triangle depicts the values of \( \binom{n}{k} \).
A Bernoulli distribution is a special case of the binomial distribution when \( N = 1 \).

For this reason, sometimes the term binomial is used to refer to a Bernoulli random variable.
Example

- Probability that a coin lands heads *at least* once during 3 flips?
Example

- Probability that a coin lands heads \textit{at least} once during 3 flips?

\[ P(X \geq 1) \]
Example

- Probability that a coin lands heads *at least* once during 3 flips?

\[
P(X \geq 1) = P(X = 1) + P(X = 2) + P(X = 3) = 0.375 + 0.375 + 0.125 = 0.875
\]