Fast Multipole Accelerated Indirect Boundary Elements for the Helmholtz Equation

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Outline

- Introduction
- Boundary Integral Equations
- Analytical Boundary Integrals
- Fast Multipole Method
- Performance study
- Some examples
- Conclusion
Introduction

- Large scale problems, $kD >> 1$, $N_{surf}=O((kD)^2)$
  - Room acoustics
  - Noise in car, aircraft, etc. interiors
  - Design of beamformers
  - Scattering off human/animal head
  - Underwater acoustics
  - More (biotechnologies, medical, etc.)
  - Cannot be handled with conventional BEM (needs acceleration)

- Indirect BEM
  - Thin plates, baffles
  - Openings
  - Simultaneous solution of the internal and external problems (materials with different acoustic properties, dispersed systems)
Helmholtz equation

\[ \Delta p(x, t) = \text{Re} \{ e^{-i\omega t} \phi(x) \}, \quad k = \frac{\omega}{C}. \]

\[ \nabla^2 \phi + k^2 \phi = 0, \quad x \in V \subset \mathbb{R}^3, \quad k \in \mathbb{R}, \]

For infinite domains (Sommerfeld radiation condition):

\[ \lim_{|x| \to \infty} \left( |x| \left( \frac{\partial \phi}{\partial |x|} - ik\phi \right) \right) = 0. \]

+ boundary conditions on the domain boundaries
Boundary Integral Equations

(Closed surfaces, for direct BEM)

Green’s identity:
\[ \pm \phi(y) = L[q] - M[\phi], \quad y \notin S, \]
\[ \pm \frac{1}{2} \phi(y) = L[q] - M[\phi], \quad y \in S, \]

Single layer potential:
\[ L[q] = \int_S q(x)G(x,y)dS(x), \]

Double layer potential:
\[ M[\phi] = \int_S \phi(x)\frac{\partial G(x,y)}{\partial n(x)}dS(x), \]

Green’s function:
\[ G(x,y) = \frac{e^{ikr}}{4\pi r}, \quad r = |x-y|. \]

Combined (Burton-Miller) BIE:
\[ \pm \frac{1}{2} [\phi(y) + \lambda q(y)] = (L + \lambda L')[q] - (M + \lambda M')[\phi], \]

Generic boundary conditions:
\[ \alpha(x)\phi(x) + \beta(x)q(x) = \gamma(x). \]

(\(\alpha, \beta, \gamma\) are given)
\[ \beta = 0: \text{Dirichlet} \]
\[ \alpha = 0: \text{Neumann} \]
\[ \alpha, \beta = \text{const: Robin} \]
Boundary Integral Equations
(Arbitrary surfaces, for indirect BEM)

Solution as a sum of single and double layer potentials:

\[ \phi(y) = L[\sigma] + M[\mu], \quad y \not\in S, \]

BIE (jump conditions):

\[ \phi^\pm(y) = L[\sigma](y) + M[\mu](y) \pm \frac{1}{2} \mu(y), \quad y \in S, \]

\[ q^\pm(y) = \mp L'[\sigma](y) \mp M'[\mu](y) + \frac{1}{2} \sigma(y), \quad y \in S. \]

+ Generic boundary conditions on each side

The problem is to determine unknown densities \( \sigma \) and \( \mu \)
Boundary Element Method

1) Discretize the surface (e.g. with a triangular mesh)

2) Compute integrals for each panel

3) Collocate BIE at the collocation points (e.g. panel centers or mesh vertices) and form a linear system of algebraic equations

4) Solve the system

5) Compute potential for arbitrary point in domain
Computation of boundary integrals

- Can be computed numerically using quadratures and special techniques to treat singularities
- Problems may appear for accurate evaluation of nearly singular, weakly singular, singular, and hypersingular integrals
- Evaluation of such integrals should be fast and robust
- We developed analytical methods
Analytical computation of boundary integrals (1)

\[ L(y) = \int_S G(|x - y|)dS(x), \quad y \in \mathbb{R}^3, \quad G(\rho) = \frac{\varepsilon^{ik\rho}}{4\pi \rho}. \]

\[ M(y) = \int_S \mathbf{n} \cdot \nabla_x G(|x - y|)dS(x), \]

\[ L'(y) = \nabla_y \int_S G(|x - y|)dS(x), \]

\[ M'(y) = \nabla_y \int_S \mathbf{n} \cdot \nabla_x G(|x - y|)dS(x). \]

Gauss divergence theorem:
Reduce surface integrals to contour integrals

\[ L(y) = \int_S \nabla_x \cdot \mathbf{F}(x, y)dS(x) = \int_C \mathbf{n}'(x) \cdot \mathbf{F}(x, y)dl(x). \]

\[ \nabla_x \cdot \mathbf{F}(x, y) = G(|x - y|), \quad \nabla_x = i_1 \frac{\partial}{\partial x_1} + i_2 \frac{\partial}{\partial x_2}. \]

\[ \mathbf{F}(x, y) = (x - y + h\mathbf{n})\frac{\varepsilon^{ik\rho} - \varepsilon^{ik|\mathbf{y}|}}{4\pi ik\rho^2}, \quad \rho = |x - y|, \quad r = |x - y + h\mathbf{n}|. \]
Analytical computation of boundary integrals (2)

Compute primitives using expansions and recursions

\[ \int_{C} \mathbf{n}^i(\mathbf{x}) \cdot \mathbf{F}(\mathbf{x}) d\mathbf{l}(\mathbf{x}) = \sum_{j=1}^{n} \int_{C_j} \mathbf{n}^i(\mathbf{x}) \cdot \mathbf{F}(\mathbf{x}) d\mathbf{l}(\mathbf{x}), \]

Line integral

\[ I_j = \int_{C_j} \mathbf{n}^i(\mathbf{x}) \cdot \mathbf{F}(\mathbf{r}) d\mathbf{l}(\mathbf{x}) = H(l_j - x', y', z') - H(-x', y', z'), \]

Primitive

\[ H(x, y, z') = \frac{-z'}{4\pi ik} \int \frac{e^{ik\rho} - e^{ik\rho'}}{x^2 + z'^2} dx, \quad \rho = \sqrt{x^2 + y^2 + z'^2} \]

Element size is small compared to the wavelength

\[ k|\rho - \rho_0| \leq k|x - x_j| \ll \pi \]

\[ H(x, y', z') = \frac{1}{4\pi ik} \left[ e^{ik\rho'} f_0(x, y', z') - e^{ik\rho_0} \sum_{l=0}^{p-1} \frac{(ik)^l}{l!} a_{p-1}(-ik\rho_0) f_l(x, y', z') \right] + O((k\Delta x)^p) \]

\[ f_0 = \text{sgn}(z') \arctan \frac{x}{|z'|}, \]

\[ f_1 = y' \text{sgn}(z') \arctan \frac{y'x}{|z'|\rho} + z' \ln|\rho + x|. \]

Other \( f_l \) computed recursively

\[ a_1(\xi) = 1, \quad a_{l+1}(\xi) = a_l(\xi) + \frac{\xi l}{l!}. \]
What the FMM does?

• Computes $N \times N$ matrix-vector product, $Ax$, for cost less than $O(N^2)$ (ideally for $O(N)$ or $O(N \log N)$);

• The catch is in the controlled accuracy (which can be machine precision, or lower for substantial speedups);

• The matrix is decomposed into sparse and dense parts, $A = A_{\text{sparse}} + A_{\text{dense}}$;

• The sparse matrix represents interaction of closely located elements (some neighborhoods); $A_{\text{sparse}}x$ can be computed in $O(N)$ operations and may require $O(N)$ memory;

• The dense matrix represents interaction of far elements (outside the neighborhoods); $A_{\text{dense}}x$ can be computed in $O(N \log^\alpha N)$ operations and requires $O(\log N)$ memory if done efficiently;

Basics of the FMM and specifics for the Helmholtz equation can be found in our book

### Standard and Fast Multipole accelerated BEM

<table>
<thead>
<tr>
<th>Task</th>
<th>Standard BEM</th>
<th>FM BEM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reformulate the problem in terms of BIE</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Discretize the boundary</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>Compute and store boundary integrals</td>
<td>Full storage, memory $\sim(kD)^4$</td>
<td>Partial storage, memory $\sim(kD)^2$</td>
</tr>
<tr>
<td>Solve linear system</td>
<td>If direct $\sim(kD)^6$, iterative $\sim N_{iter} (kD)^4$</td>
<td>Iterative $\sim N_{iter} (kD)^2$, efficient FMM preconditioner</td>
</tr>
</tbody>
</table>

Max solvable problem size (PC): $N \sim 3 \cdot 10^4 (kD \sim 10^2) \quad N \sim 3 \cdot 10^6 (kD \sim 10^3)$
Validation for Sphere and Disk

Sphere, \( N = 2048 \)

Disk, \( N = 1049 \)
Sphere performance

Trading memory for speed

**Table 1**: Running times and memory usage for different mesh sizes and values of $kD$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$kD$</th>
<th>$t_1$ (s)</th>
<th>$t_2$ (s)</th>
<th>$t_3$ (s)</th>
<th>mem$_1$ (GB)</th>
<th>mem$_2$ (GB)</th>
<th>mem$_3$ (GB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>50700</td>
<td>83.4</td>
<td>$1.29 \times 10^3$</td>
<td>354</td>
<td>193</td>
<td>0.31</td>
<td>1.00</td>
<td>5.22</td>
</tr>
<tr>
<td>101568</td>
<td>118</td>
<td>$3.19 \times 10^3$</td>
<td>592</td>
<td>411</td>
<td>0.62</td>
<td>1.95</td>
<td>5.58</td>
</tr>
<tr>
<td>202800</td>
<td>167</td>
<td>$2.06 \times 10^4$</td>
<td>$4.59 \times 10^3$</td>
<td>N/A</td>
<td>1.02</td>
<td>3.54</td>
<td>N/A</td>
</tr>
<tr>
<td>401868</td>
<td>235</td>
<td>N/A</td>
<td>$4.59 \times 10^3$</td>
<td>N/A</td>
<td>N/A</td>
<td>7.12</td>
<td>N/A</td>
</tr>
</tbody>
</table>

6 elements per wavelength
Error in solution $\sim 1.5\%$
Timing and Memory Usage Data

\[ k = CN^{1/2} \]

(\sim 20 \text{ elements per wavenegth})
Rewrite Helmholtz equation in oblate spheroidal coordinates.

A disk can be represented in oblate spheroidal coordinates as the isosurface, $\xi = 0$.

Expand scattered field in terms of oblate spheroidal wave functions

$$\phi^{\text{scat}} = -2 \sum_{n=0}^{\infty} \frac{i^n}{N_{0n}} S_{0n} (-ic, -1) \frac{R_{0n}'(-ic, i0)}{R_{0n}'(-ic, i0)} S_{0n} (-ic, \eta) R_{0n}^{(3)} (-ic, i\xi)$$
Validation: Disk

Analytical Sol. (Real Comp.) along Surface of Disk

Numerical Sol. (Real Comp.) along Surface of Disk

Error (Real Comp.) along Surface of Disk
Validation: Disk

Disk, $k = 10$, Analytical Solution

Disk, $k = 10$, Numerical Solution

Disk, $k = 10$, Percent Error
Example 1: Simulations of scattering from a parabolic antenna and a sphere
Example 2: Computation of acoustic bidirectional reflectance distribution function (BRDF)

Sinusoidal surface
(Case of Sakuma et al, 2009)

diameter = 2.655 m,
amplitude= 0.0256 m.
period = 0.177 m

Incident wave:

\[ f = 2 \text{ kHz}, \]
\[ \theta' = 31.5^\circ, \]
\[ \phi' = 181.5^\circ. \]
Conclusion

- Analytical formulae for boundary integrals are developed and tested. That can be used in any direct or indirect BEM.

- A fast multipole accelerated indirect boundary element method for the Helmholtz equation in 3D is developed and tested.

- The FMM acceleration and memory reduction enables indirect BEM solution with $\sim 10^5$-$10^6$ elements on a contemporary multicore PCs.

- More work is needed for efficient FMI BEM including hardware acceleration (e.g. graphics processors) and algorithms... this is the subject of our ongoing work.
THANK YOU!