I. INTRODUCTION

A liquid with bubbles represents a classical example of nonlinear medium, which manifests a variety of very rich physical phenomena associated with its acoustical response. The behavior of the bubbles in the prescribed acoustic fields, as well as the propagation of acoustic waves in a bubbly medium, are well studied experimentally and numerically (e.g., Refs. 1–6). Using terminology that is accepted in nonlinear dynamics, such studies could be referenced as the cases of a one-way field-particle interaction, because the bubbles are driven by the pressure field although they do not modify the field itself or the field does not modify the acoustic properties of the medium.

However, bubbles oscillating in an acoustic field should alter the media and change its physical characteristics like the volume fraction, the bubble size and, consequently, the compressibility. That represents, in fact, a case of a two-way field-particle interaction, which was understood long ago. Probably, for the first time, the tendency to sound self-focusing due to the cavitation inception was considered theoretically in Ref. 7 and observed experimentally in Ref. 8. These findings prompted theoretical studies of nonlinear acoustic effects associated with collective dynamics of bubbles employing analogy with nonlinear optics.9–11 The second phenomenon which attracted the attention of researchers was the experimental observation that the bubbles formed due to the acoustic cavitation inception group themselves in remarkable branched structures of filaments called “streamers.”12,13 The whole patterns were called “acoustic Lichtenberg figures” because of the visual similarity with the electrical discharge patterns observed centuries ago by Lichtenberg.14

A two-way field-particle interaction in cavitating liquids occurs due to three major phenomena: (1) an inception of bubbles in an otherwise uniform acoustically irradiated liquid;15 (2) a very slow change of the bubble size due to rectified diffusion;16 and (3) a relatively slow motion of bubbles driven by the acoustic radiation forces for bubbles known as the Bjerknes forces.17 Several attempts have been made to model the two-way field-particle interaction in cavitating liquids accounting for all three phenomena and using both continuum and particle models.14,18–24 Despite numerical simulations showing bubble patterns that qualitatively agree with experimental observations, substantial improvement of the models is needed to achieve better results.

Most of the uncertainty comes from modeling of cavitation inception kinetics. To bypass this difficulty, experiments were recently conducted to observe a water-air bubbly liquid exposed to an acoustic field of intensity below the cavitation threshold.25–27 It was shown that bubbles can be “pushed” away from the acoustic source forming a region almost free from bubbles—the phenomenon termed as the acoustically induced transparency (AIT). It was also shown that the observed effect cannot be explained by well-known theories of single bubble drifting in an acoustic field. This is a collective effect of bubbles, which can be described by a mathematical model of bubble organization in an acoustic field.28–27 At the same token, it is well known from the experiments on “acoustic Lichtenberg figures”14 and from systematic studies on the dynamics of bubbles in a standing sound field, that in fresh tap water containing some submerged microbubbles at very low pressure amplitudes the bubbles move toward the pressure antinodes forming bubble clusters.

Thus, there is a need for a systematic theory of self-organization of bubbly liquid that covers both of these phenomena.
effects: the clustering around pressure antinodes and the AIT. This paper presents such a theory. The goal is to put the theory and numerical methods in a solid theoretical framework, which can start from a weakly nonlinear theory of self-organization based on the equations of continuum mechanics of multiphase systems for diluted disperse systems. Such a modeling is an important step for further theoretical developments, which should take into account higher volume fractions of bubbles and strong nonlinearity. In addition, we believe that the theory should start from a one-dimensional case, to provide the insight for future developments. This approach contrasts with attempts to jump immediately to mathematical modeling of a complex situation realized in experiments, where we have waves of large amplitude, regions of high bubble concentration, and substantially three-dimensional (3D) structures.

II. MODEL

A. Bubble dynamics in an acoustic field

Assume that bubbles are spherical and experience small amplitude oscillations, collisions are negligible, and mass diffusion in the liquid plays a minor role (particularly, we neglect the rectified diffusion). For time-harmonic acoustic fields of circular frequency \( \omega \), we assume a harmonic bubble response

\[
p = p_0 \left[ 1 + \epsilon \text{Re} \{ A e^{-i \omega t} \} \right],
\]
\[
a = a_0 \left[ 1 + \epsilon \text{Re} \{- A(a_0) e^{-i \omega t} \} \right],
\]

where \( p \) and \( p_0 \) are the total and static pressure, \( \epsilon \) the relative pressure amplitude, \( a \) and \( a_0 \) are the bubble radius and its period average, \( A \sim 1 \) is the normalized complex amplitude of the pressure at the bubble location, and \( A(a_0) \) is the bubble response function. This function for a bubble in an infinite liquid can be found in Ref. 3,

\[
A(a_0) = \frac{a_0^2}{a_0^2 - a_0^2 - i \eta},
\]
\[
a^2_\sigma = \left[ 3 \gamma + \frac{2 \sigma}{p_0 a_0} (3 \gamma - 1) \right] a_0^2, \quad a^2_\varepsilon = \frac{p_0}{\omega^2 \rho_l},
\]

where \( a_\sigma \) and \( a_\varepsilon \) are the resonance radius of a single bubble and the characteristic bubble length scale, \( \sigma \) and \( \rho_l \) are the surface tension and the liquid density, and \( \gamma(a_0) \) and \( \eta(a_0) \) are the effective polytropic exponent and the dissipation coefficient,

\[
\eta = a^2_\varepsilon \left[ \frac{4 \mu l_{\text{eff}}}{p_0} + \delta_a a_0^3 a^2_\varepsilon - i \text{Im} \left\{ \Theta \left( \frac{i \omega a_0^2}{K_g} \right) \right\} \right],
\]
\[
\gamma = \frac{1}{3} \text{Re} \left\{ \Theta \left( \frac{i \omega a_0^2}{K_g} \right) \right\},
\]
\[
\Theta(\xi) = \frac{3 \gamma g^\prime}{\xi + 3 (g^\prime - 1) \left( \xi^{1/3} \coth \xi^{1/2} - 1 \right)}; \quad \delta_a = \frac{p_0}{\rho_l C^2},
\]

Here \( \mu \) and \( C_l \) are the liquid viscosity and the sound speed, while \( K_g \) and \( g \) are the gas thermal diffusivity and the adiabatic exponent.

As the bubble oscillates, it also experiences a period-averaged displacement due to the acoustic radiation and external forces, such as the gravity. This motion occurs in a slow time scale, \( t_s \). As we present here all equations in a dimensional form, the units in which the slow and the fast times are measured are the same (seconds). So, such a two-scale splitting of the time has a physical meaning mentioned above, but formally, we do not introduce a small parameter related to the ratio of the period of the acoustic field to the characteristic time of the bubble drift. Some estimates validating this assumption are provided in Refs. 24 and 27. The momentum conservation equation accounting for the added mass, Bjerknes, quasi-steady viscous drag, and buoyancy forces can be written in the form

\[
\frac{d \mathbf{v}_0}{d t_s} = \frac{3 p_0 c^2}{\rho_l} \text{Re} \{ A(a_0) \bar{A} \} - 2g - \frac{1}{\tau_\mu} \mathbf{v}_0,
\]
\[
\tau_\mu = \frac{p_0 \sigma_0^2}{18 \kappa_{\mu} \mu_l}.
\]

Here \( \mathbf{v}_0 \) is the period-averaged (“drift”) velocity, the bar denotes the complex conjugate, \( g \) is the gravity acceleration, \( \tau_\mu \) is the relaxation time due to viscosity, and \( \kappa_\mu \) is the viscous drag coefficient, which for bubbles can be approximated as

\[
k_\mu = \frac{1 + 0.9 \text{Re}_b^{1/2}}{3 + 0.3 \text{Re}_b^{1/2}}, \quad \text{Re}_b \sim 1 - 2.2 \text{Re}_b^{1/2}. \tag{5}
\]

We derived this approximation simply by matching the low Reynolds number limit, \( k_\mu = 1/3 \), by assuming that \( k_\mu \) is a rational function of the relative thickness of the viscous boundary layer, \( \text{Re}_b^{1/2} \), and by matching Moore’s asymptotics at large Reynolds numbers \( k_\mu \sim 1 - 2.2 \text{Re}_b^{1/2} \). We do not insist on this form of approximation, understanding that this is a model, which misses some other forces (e.g., the history Basset-type force), while at high Strouhal numbers for spherical bubbles \( k_\mu \rightarrow 1 \) independently on the Reynolds numbers.

In many practically interesting situations, the characteristic times of self-organization are much larger than the relaxation time \( \tau_\mu \) (see Ref. 27). In this case the unsteady added mass force can be neglected, and Eq. (4) reduces to

\[
\mathbf{v}_0 = \tau_\mu \left( \frac{3 p_0 c^2}{\rho_l} \text{Re} \{ A(a_0) \bar{A} \} - 2g \right). \tag{6}
\]

We provide a more accurate theoretical estimate, as we introduce the characteristic time for the process [see Eq. 19 below].

If the acoustic field and its gradient are known at any bubble location, then the kinematic equation for the period-averaged bubble position, \( \mathbf{r}_0 \),

\[
\frac{d \mathbf{r}_0}{d t_s} = \mathbf{v}_0,
\]

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closes the model. Particularly, if the bubbles do not affect the acoustic field (the one-way interaction) the above equations can be used to study the linear and weakly nonlinear bubble dynamics in acoustic fields.

**B. Acoustics of bubbly liquids**

In the present study we accept a continuum model of a multiphase system (e.g., Ref. 4). In this case the acoustic field in a diluted mixture (the volume fraction of bubbles \( \varepsilon \ll 1 \)) is described by the Helmholtz equation

\[
\nabla^2 A + k^2(r, t)A = 0, \quad k^2 = k_f^2 + k_b^2,
\]

where \( k_f = \omega/C_f \) is the wave number in the liquid without bubbles, \( k \) is the complex wave number for the mixture, and \( k_b \) describes the contribution of the bubbles to the total wave number. The latter may vary in space (and in the slow time scale) due to the non-uniformity of the bubble distribution. The dispersion relationship for bubbly liquids was derived and analyzed by several researchers. In the present model, we use the result of Ref. 3 for polydisperse systems

\[
\begin{align*}
\frac{\kappa_0^2(r, t)}{\kappa_0^2(r, t)} &= \frac{4\pi \rho_f \omega^2}{\rho_p} \int_0^\infty N_b(a_0', r, t_0) \Lambda(a_0') da_0', \\
\omega_0(r, t_0) &= \frac{4}{3} \pi \int_0^\infty N_b(a_0', r, t_0) a_0'^3 da_0',
\end{align*}
\]

where \( N_b(a_0', r, t_0) \) is the bubble size distribution function, or the distribution density \( (N_b da_0' \) is the number of bubbles of undisturbed sizes between \( a_0' \) and \( a_0' + da_0' \) per unit volume in an elementary volume of the mixture centered at \( r \) at time \( t_0 \), and \( \omega_0 \) is the period-averaged bubble volume fraction. Efficient modeling of polydisperse systems can be done using \( M \) fractions of bubbles of different size \( a_{j0} \) and the number density \( n_{j0}, j = 1, \ldots, M \). Formally, we have Eq. (9) where the distribution density is a sum of Dirac’s delta-functions with respective weights,

\[
\begin{align*}
\frac{\kappa_0^2(r, t)}{\kappa_0^2(r, t)} &= \frac{3\rho_f \omega^2}{\rho_p} \sum_{j=1}^M n_{j0}(r, t_0) \Lambda(a_{j0}), \\
N_b(a_0', r, t_0) &= \sum_{j=1}^M n_{j0}(r, t_0) \delta(a_0' - a_{j0}), \\
\omega_0 &= \frac{4}{3} \pi a_0^3 n_{j0}(r, t_0), \quad j = 1, \ldots, M, \quad n_{j0} = \sum_{j=1}^M n_{j0}.
\end{align*}
\]

At \( M = 1 \) we have the case of monodisperse mixtures.

**C. Closed model of self-organization**

Summarizing the framework of the continuum model for a polydisperse bubbly liquid containing \( M \) bubble fractions, we have the following closed system of equations describing the two-way medium-field interaction,

\[
\begin{align*}
\nabla^2 A + k^2 A &= 0, \quad k^2 = k_f^2 + k_b^2, \\
k_b^2 &= \frac{3\rho_f \omega^2}{\rho_p} \sum_{j=1}^M n_{j0} \Lambda(a_{j0}), \\
\frac{\partial \varphi}{\partial t} + \nabla \cdot (\varphi \mathbf{v}) &= 0, \\
\mathbf{v}_j &= \frac{3\rho_f \omega^2}{\rho_p} \Re \left\{ \Lambda(a_{j0}) A \nabla A \right\} - 2g, \quad j = 1, \ldots, M.
\end{align*}
\]

Here the continuity equations for each fraction replace Eq. (7) providing tracking of each bubble. Note that in the case of the one-way interaction (a few bubbles in the acoustic field, no feedback) the Bjerknes force acting on a bubble sometimes is decomposed into the primary (bubble-field) and the secondary (bubble-bubble) Bjerknes forces. In the present model, all bubbles interact via the acoustic field. So the Bjerknes force here accounts for the both types of the Bjerknes force.

This nonlinear system describing the spatiotemporal evolution of the bubbles and the acoustic field can be integrated subject to initial conditions for the bubble spatial distribution and boundary conditions for the Helmholtz equation. In the present study, we consider one-dimensional problems to identify anticipated regimes and effects.

Assume that the reference frame is selected in the way that \( g = -\frac{i}{2} \omega \mathbf{g} \) and all spatial distributions depend on only the \( z \)-coordinate. Assume also that the liquid occupies region \( 0 \leq z \leq H \). At \( z = 0 \), we have either free surface (sound soft boundary) or a contact with a solid surface (sound hard or impedance boundary), while an acoustic transducer is located at \( z = 0 \) and its surface oscillates with amplitude \( \Delta \), means \( \Delta z = \Delta \text{Re} \{ e^{-i\omega t}\} \). The velocity of the liquid adjacent to the surface then can be determined as

\[
\frac{\partial \mathbf{v}}{\partial t} \bigg|_{z=0} = \frac{d \Delta z}{dt} = \Delta \text{Re} \{ -i\omega e^{-i\omega t}\}.
\]

We have from the momentum conservation equation,

\[
\frac{\partial \rho_f}{\partial t} \bigg|_{z=0} = -\rho_f \frac{\partial \mathbf{v}}{\partial t} \bigg|_{z=0} = \rho_f \omega^2 \Delta \text{Re} \{ e^{-i\omega t}\}.
\]

Hence, if the pressure field is described by Eq. (1) and we define dimensionless parameter \( \epsilon \) as

\[
\epsilon = \frac{\rho_f \omega^2}{\rho_k k_f} \Delta = \frac{\rho_f C_f}{\rho_p} \Lambda,
\]

we obtain the following boundary conditions for the acoustic amplitude,

\[
\frac{\partial A}{\partial \xi} \bigg|_{z=0} = k_f, \quad A \bigg|_{z=H} = 0.
\]

The boundary condition at the free surface can be replaced by \( \partial A/\partial \xi \bigg|_{z=H} = 0 \) for sound hard boundary or by the Robin conditions for the impedance boundary. The initial conditions can be selected as
\[ \varepsilon_0 |_{t_* = 0} = \varepsilon_{00}, \quad j = 1, \ldots, M. \]  

In the present study we assume a spatially uniform initial distribution of the bubbles, while, generally, one can set \( \varepsilon_0 = \varepsilon_{00}(r) \).

Further, we rewrite equations in a dimensionless form and identify the main dimensionless parameters. We propose to use the following variables, parameters, and scales,

\[ \zeta = \frac{z}{L_*}, \quad \tau = \frac{t}{t_*}, \quad K = kL_*, \quad \eta_j = \frac{v_{j0}}{U_*}, \quad \beta_j = \frac{\varepsilon_{j0}}{\varepsilon_{00}}, \]

\[ m_j = \frac{3C_l^2 \rho_f \rho_0 \varepsilon_{00}^2}{\rho_s U_*} |\lambda(a_0)|, \quad \epsilon_j = \frac{\lambda(a_0)}{|\lambda(a_0)|}, \quad h = \frac{H}{L_*}, \]

\[ c_j = \frac{a_j^2|\lambda(a_0)|}{\lambda(a_j)}, \quad g_j = \frac{\rho_f g a_j^2 \varepsilon_{00}^2}{3 \mu_i U_*}, \quad j = 1, \ldots, M, \]

\[ \Re_{bj} = k_{bj}|u_j|, \quad k_{bj} = \frac{2\rho_f a_j^2 \varepsilon_{00}^2}{\mu_i}, \]

\[ L_* = \frac{1}{k_j}, \quad U_* = \frac{v_j a_j^2 k_j}{2 \mu_i} |\lambda(a_j)|, \quad t_* = \frac{L_*}{U_*}, \quad j = 1, \ldots, M. \]  

where \( a_j \) is some representative bubble radius, \( \varepsilon_{00} \) is the total volume fraction of bubbles. The dimensionless 1D equations with boundary and initial conditions then follow from Eqs. (11), (15), and (16),

\[ \frac{\partial \beta_j}{\partial \tau} + \frac{\partial (\beta_j \eta_j)}{\partial \zeta} = 0, \]

\[ u_j = \frac{1}{3k_\mu(\Re_{bj})} \left( c_j \Re \left( \frac{\epsilon_j^2 A}{\partial \zeta} \right) + g_j \right), \]

\[ \Re_{bj} = k_{bj}|u_j|, \quad j = 1, \ldots, M, \]

\[ \frac{\partial^2 A}{\partial \zeta^2} + K^2 A = 0, \quad K^2 = 1 + \sum_{j=1}^{M} \beta_j m_j \epsilon_j^2, \]

\[ \frac{\partial A}{\partial \zeta} \bigg|_{\zeta = 0} = 1, \quad A|_{\zeta = h} = 0, \quad \beta_j |_{\zeta = 0} = \beta_{j0}, \quad \sum_{j=1}^{M} \beta_j \varepsilon_{00} = 1. \]  

(18)

The time scale \( t_* \), Eq. (17), can be compared with the viscous relaxation time \( \tau_\mu \), Eq. (4). For bubbles of characteristic size \( a_* \), we have

\[ \tau^{*} = \frac{t_*}{\tau_\mu} = \frac{\varepsilon_0}{\epsilon} \left( \frac{\rho_f a_*^{2} \varepsilon_0}{\mu_i} \right)^{2} |\lambda(a_*)|. \]  

(19)

Formally, in a weakly nonlinear theory \( \epsilon \) is asymptotically small, and we have \( \tau_\mu^{*} = O(\epsilon^2) \), which justifies simplification (6). Note also that the coefficient near \( \epsilon^2 \) is on the order of the unity for acoustic frequency \( \sim 100 \) kHz and water in room conditions containing non-resonant air bubbles of radius \( \sim 20 \) \( \mu \)m.

### III. ANALYTICAL STUDY
#### A. Simplified equations

For the analytical study provided below we accept a few more simplifications, namely, (a) the mixture is monodisperse \( (M = 1) \), (b) the effect of gravity is negligible \( (g_1 \ll 1) \), (c) the drag coefficient \( k_\mu \), is constant. Equations (18) then reduce to

\[ \frac{\partial \beta}{\partial \tau} + \frac{\partial (\beta u)}{\partial \zeta} = 0, \quad u = \frac{1}{3k_\mu \Re} \left( \frac{\epsilon^2 A \partial A}{\partial \zeta} \right), \]

\[ \frac{\partial^2 A}{\partial \zeta^2} + K^2 A = 0, \quad K^2 = 1 + \beta \eta, \]

\[ \frac{\partial A}{\partial \zeta} \bigg|_{\zeta = 0} = 1, \quad A|_{\zeta = h} = 0, \quad \beta |_{\zeta = 0} = 1, \]  

(20)

where we dropped subscript \( j = 1 \) serving for identification of fractions. Note also that the dimensionless velocity can be rewritten as

\[ u = \frac{1}{3k_\mu \beta m} \Re \left\{ \beta \eta \epsilon^2 A \frac{\partial A}{\partial \zeta} \right\}, \]

\[ = \frac{1}{3k_\mu \beta m} \Re \left\{ (K^2 - 1) A \frac{\partial A}{\partial \zeta} \right\}, \]

\[ = \frac{1}{3k_\mu \beta m} \Re \left\{ - \frac{1}{A} \frac{\partial^2 A}{\partial \zeta^2} - 1 \right\} \frac{\partial A}{\partial \zeta} \right\}, \]

\[ = - \frac{1}{6k_\mu \beta m \partial \zeta} \left[ A^2 + \left| \frac{\partial A}{\partial \zeta} \right|^2 \right]. \]  

(21)

It is remarkable that only four parameters \( m, \eta, h, k_\mu \) control the dynamics of the system, which is valuable for parametric studies. In fact, \( k_\mu \) simply controls the time and velocity scales and can be excluded from Eq. (20) by setting \( w = (3k_\mu \eta) u, \) and \( \theta = \tau/3k_\mu \). In this case equations in variables \( w \) and \( \theta \) will be the same as (20) at \( k_\mu = 1/3, i.e., at low \)Reynolds numbers. Hence, \( m, \eta, h, \) and \( k_\mu \) are regime controlling parameters. Equation (2) provides \( 0 < \eta < \pi \) since \( \eta > 0 \). Also by definition, \( m > 0 \) and \( h > 0 \).

#### B. Initial stage

Even simplified case (20) is too complicated to obtain an exact analytical solution. So let us consider the initial stage.

For initially homogeneous mixture, we have spatially uniform \( K = K_0 \). In this case, the general solution of the Helmholtz equation is

\[ A|_{\zeta = 0} = C_1 e^{iK_0 \zeta} + C_2 e^{-iK_0 \zeta}, \]

\[ K_0 = (1 + \eta^2)^{1/2} = K_{00} + iK_{00}, \quad K_{00} > 0, \quad K_{00} > 0. \]  

(22)

This shows that \( K_{00} \) and \( K_{00} \) can be used instead of \( m \) and \( \eta \) due to one-to-one mapping of region \( K_{00} > 0, K_{00} > 0 \) in space \( (K_{00}, K_{00}) \) to \( m > 0, 0 \leq \eta \leq \pi \) in space \( (m, \eta) \). The integration constants can be determined from the boundary conditions Eqs. (20),

\[ C_1 = \frac{1}{iK_0} \frac{1}{1 + e^{2iK_0 h}}, \quad C_2 = -\frac{1}{iK_0} \frac{1}{1 + e^{2iK_0 h}}. \]  

(23)

Using Eq. (21) we can determine the bubble velocity distribution
In the case of substantial dissipation all bubbles move away from the source of sound; particularly, in the limit $K_0 h \to \infty$ we have only an outgoing wave in the region adjacent to the transducer and
\[
u|_{r=0} \sim \left(1 + \left|K_0^0\right|^2\right) K_0 e^{-2K_0^0} > 0, \quad K_0(h - \zeta) \to \infty.
\] (27)

This means that in this case all bubbles in the system (except a relatively small region near $\zeta = h$) move in the positive direction independently on the bubble size. We call this regime as AIT. So this is a different mode of system dynamics.

In the intermediate range of parameter $K_0(h - \zeta)$ a more complex regime combining the two limiting cases can be observed. Indeed, near the transducer ($\zeta \ll h$) the value of $K_0 h$ can be large enough to provide the AIT mode, while at $\zeta \approx h$ the value of $K_0(h - \zeta)$ can be substantially small to provide the clustering mode. This is also seen in Fig. 1. According to Eq. (24) the zeros of bubble velocity can be controlled by parameters $\vartheta$ and $K_0$ besides $K_0(h - \zeta)$. The dependence on $K_0$, in fact, can be removed if we consider a sufficient condition for the realization of the wave of transparency. Indeed, the following condition
\[
1 - y^2 - 2|\vartheta| y > 0, \quad y = e^{-2K_0(h - \zeta)}
\] (28)
is sufficient to have $u|_{r=0} > 0$. If we change the sign from $>$ to $<$ in Eq. (28), then we obtain a necessary condition for clustering, i.e., existing of regions $u|_{r=0} < 0$. It is not difficult to see that condition (28) is equivalent to
\[
K_0(h - \zeta) > \frac{1}{2} \ln \frac{1}{\sqrt{\vartheta^2 + 1 - |\vartheta|}} \left(\vartheta < \sinh[2K_0(h - \zeta)]\right).
\] (29)

FIG. 1. The initial dimensionless bubble velocity, $u$, and the normalized velocity, $u_{nor}$, in bubbly liquids with different attenuation coefficients.

---

C. Two modes of self-organization

Figure 1 shows typical velocity profiles $u|_{r=0}(\zeta)$ at different values of the attenuation coefficient $K_0$. It is seen that at relatively small values of $K_0$ function $u|_{r=0}$ and, respectively, a normalized function $u_{nor} = (ue^{2K_0} / u|_{r=0})|_{r=0}$ may have many zeros. At the increasing $K_0$ the number of the zeros reduces, and they occur near $\zeta = h$. At large $K_0$, the only zero is $\zeta = h$ (which is a zero at any values of parameters due to the boundary condition). Hence, at low dissipation we have a case when the bubbles should demonstrate clustering. Particularly, in the limit $K_0 h \to 0$ we have a standing wave and
\[
u|_{r=0} \sim \frac{2K_0(1 - |K_0|^2)}{3k_0 m |K_0|^2 [1 + e^{2K_0 h}]} \sin[2K_0 h (h - \zeta)],
\]
\[K_0(h - \zeta) \to 0.
\] (26)

Note that the sign of $1 - |K_0|^2$ depends on the bubble size and at small concentrations, $m \ll 1$, it is negative for sub-resonant bubbles and positive for super-resonant bubbles, which should cluster near the pressure antinodes and nodes, respectively, and which is a well-known fact for single bubbles.

\[
u|_{r=0} \sim \frac{1 + |K_0|^2}{3k_0 m |K_0|^2[1 + e^{2K_0 h}]} \left[1 - e^{-4K_0 h - \zeta} \right] + 2\vartheta e^{-2K_0 h - \zeta} \sin[2K_0 h (h - \zeta)]\right)\right),
\] (24)

where
\[
\vartheta = \frac{K_0(1 - |K_0|^2)}{K_0(1 + |K_0|^2)}.
\] (25)

---

\[
u|_{r=0} = \frac{1 + |K_0|^2}{3k_0 m |K_0|^2[1 + e^{2K_0 h}]} e^{-2K_0 h - \zeta} > 0, \quad K_0(h - \zeta) \to \infty.
\] (27)

This means that in this case all bubbles in the system (except a relatively small region near $\zeta = h$) move in the positive direction independently on the bubble size. We call this regime as AIT. So this is a different mode of system dynamics.

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is sufficient to have $u|_{r=0} > 0$. If we change the sign from $>$ to $<$ in Eq. (28), then we obtain a necessary condition for clustering, i.e., existing of regions $u|_{r=0} < 0$. It is not difficult to see that condition (28) is equivalent to
\[
K_0(h - \zeta) > \frac{1}{2} \ln \frac{1}{\sqrt{\vartheta^2 + 1 - |\vartheta|}} \left(\vartheta < \sinh[2K_0(h - \zeta)]\right).
\] (29)
Regarding regime classification, it can be noticed that even though the clustering mode can formally exist near the end \( \zeta = h \) at large enough \( K_{\text{fl}} \), the velocity can be substantially small due to its exponential decay with \( \zeta \) (compare graphs for \( u \) and \( u_{\text{near}} \) in Fig. 1). Practically, it does not matter whether the velocity zeros exist near \( \zeta = h \) or not (one also should keep in mind that in this case the neglected buoyancy forces can exceed the acoustic radiation forces). So, for the overall characterization of the system behavior in the space of parameters, we accept a convention that if all bubbles located at \( \zeta < \pi/K_{\text{fl}} \) (the half of the wavelength; also assume \( h > \pi/K_{\text{fl}} \)) move away from the transducer then we have the AIT mode, otherwise we have the clustering mode. The distance \( \pi/K_{\text{fl}} \) is equal to one period of the spatial oscillations of the initial velocity [see Eq. (24)]. Using Eq. (29), we obtain the following sufficient condition for the realization of the AIT mode

\[
F(K_{\text{fl}}, K_{\text{fl}}) < h, \quad F = \frac{\pi}{K_{\text{fl}}} + \frac{1}{2K_{\text{fl}}} \ln \frac{1}{\sqrt{\vartheta^2 + 1} - |\vartheta|}. 
\] (30)

This shows that the boundaries separating one mode from the other are nothing but the levels of function \( F \). Figure 2 shows such levels. Also, function \( F \) can be considered as a function of two arguments \( m \) and \( \vartheta \) [see Eq. (20)]. The levels of function \( F \) for these arguments are displayed in Fig. 2 as well. Note that \( \vartheta = \pi/2 \) corresponds to resonance bubbles, which at low \( m \) results in \( K_r = 1 \). It is also seen that at large enough \( m \) and \( h \) (\( h = 2\pi \) corresponds to the wavelength of sound in the liquid without bubbles) the AIT regime dominates, while for small \( m \) it can be observed only for nearly resonant bubbles; the clustering mode prevails for bubbles of a different size. Figure 3 illustrates function \( F \), as a function of two dimensionless parameters \( z_{00} \) and \( a_M a_{\text{fl}} \), where \( a_M = a_0 \sqrt{3/\gamma} \) is the Minnaert resonance radius [which is close, but a bit different from the “true” resonance radius \( a_r \) in Eq. (2)]. Calculations are shown for water-air systems at \( f = 100 \text{ kHz} \). However, we also performed computations in the frequency range \( f = 10 \text{ kHz} \) to \( 1 \text{ MHz} \) and found that the map is close enough to what is shown in Fig. 3. It is clearly seen that at low void fractions the AIT mode can be observed only for bubbles about the resonance size providing the strongest dissipation. The range substantially increases and may form some interesting patterns in the parametric space at larger \( z_{00} \).

D. Formation of waves of AIT

In the AIT regime the bubble velocity at \( \tau = 0 \) is finite and positive. So, at \( \tau > 0 \) near \( \zeta = 0 \) there should appear a
region of liquid free of bubbles ("pure liquid"). This means that the bubble volume fraction should have a discontinuity propagating as a shock wave of the void fraction (such as shown in Fig. 6).

Let us consider the limiting case \( K_\beta h \to \infty \). Denote the coordinate of the front of the shock wave as \( \zeta(\tau) \) and mark all parameters on the front with subscript \( f \). At small times the bubble concentration in region \( \zeta \geq \zeta_f \) does not change significantly, so we can assume that \( K \) in this region is close to constant and rewrite Eqs. (22) and (24) in the form

\[
A = A_f e^{K_\beta (\zeta - \zeta_f)}, \quad \frac{\partial A}{\partial \zeta} = iK_\beta A_f e^{K_\beta (\zeta - \zeta_f)},
\]

\[
u = \frac{|A_f|^2}{\rho m} (1 + |K_\beta|^2) K_\alpha e^{-2K_\alpha (\zeta - \zeta_f)}, \quad \zeta \geq \zeta_f. \tag{31}
\]

In the pure liquid region, we have \( \beta = 0 \). Therefore, the solution of the Helmholtz equation satisfying the boundary condition at \( \zeta = 0 \) can be written in the form

\[
A(\zeta, \tau) = \sin \zeta + D(\tau) \cos \zeta. \tag{32}
\]

Note then that the density of a diluted mixture is almost the same as the density of the carrier liquid, which provides continuity of both \( A \) and \( \partial A/\partial \zeta \) across the shock wave. Thus, we can match solutions (32) and (31) at \( \zeta = \zeta_f \):

\[
\sin \zeta_f + D \cos \zeta_f = A_f, \quad \cos \zeta_f - D \sin \zeta_f = iK_\beta A_f. \tag{33}
\]

Solving this system with respect to \( D \) and \( A_f \), we obtain

\[
A_f = \frac{1}{\sin \zeta_f + iK_\beta \cos \zeta_f}, \quad D = \frac{\cos \zeta_f - iK_\beta \sin \zeta_f}{\sin \zeta_f + iK_\beta \cos \zeta_f}. \tag{34}
\]

From Eq. (31) and the kinematic condition, we determine the dynamics of the wave front,

\[
u_f = \frac{(1 + |K_\beta|^2) K_\alpha}{3k_\beta m |\sin \zeta_f + iK_\beta \cos \zeta_f|^2}, \quad \frac{d\zeta_f}{d\tau} = \nu_f. \tag{35}
\]

These equations can be integrated analytically. Indeed, we have

\[
\tau = \int_0^{\zeta_f} \frac{d\zeta}{\nu} = \frac{3k_\beta m}{(1 + |K_\beta|^2) K_\alpha} \int_0^{\zeta_f} \sin \zeta + iK_\beta \cos \zeta_f^2 d\zeta
\]

\[
= \frac{3k_\beta m}{2(|K_\beta|^2 + 1) K_\alpha} \left[ (|K_\beta|^2 + 1) \zeta_f + (|K_\beta|^2 - 1) \sin \zeta_f \cos \zeta_f - 2K_\alpha \sin^2 \zeta_f \right]. \tag{36}
\]

Consider now at which times this asymptotic solution can be used. The only constraint we have is that the bubbly medium is homogeneous, i.e., \( \beta \approx \beta_0 = 1 \). The maximum change of \( \beta \) occurs at the wave front. Hence, we have at small times from Eqs. (20), (31), and (34)

\[
\frac{d\beta_f}{d\tau} = \frac{\partial \beta_f}{\partial \zeta} = \frac{\partial \beta_f}{\partial \zeta}\bigg|_{\zeta=\zeta_f} \sim 2K_\alpha \nu_f
\]

\[
\sim 2K_\alpha \frac{d\zeta_f}{d\tau}. \tag{37}
\]

So, the solution should be valid at

\[
2K_\alpha \zeta_f \ll 1. \tag{38}
\]

At \( K_\alpha \gg 1 \) this can be realized only at \( \zeta_f \ll 1 \), and Eq. (36) provides the times at which the small time approximation is valid,

\[
\tau \ll \frac{3k_\beta m |K_\alpha|^2}{2K_\alpha^2 (|K_\alpha|^2 + 1)} = \left( \frac{3k_\beta m |K_\alpha|^2}{2K_\alpha^2 (|K_\alpha|^2 + 1)} \right). \tag{39}
\]

Note that if the dissipation is small, \( K_\alpha^2 \ll |K_\beta|^2 \), the right-hand side can be large enough, so the "small times" are not necessarily \( \tau \ll 1 \). More general condition (38) shows that the approximation is valid when the distance traveled by the shock wave is smaller than the characteristic length of the attenuation of the acoustic waves. The continuity equation shows that \( \beta \) exponentially decays in space,

\[
\frac{\partial \beta_f}{d\tau} = -\frac{\partial (\beta u_f)}{\partial \zeta} = 2 \frac{|A_f|^2}{m} (1 + |K_\beta|^2) K_\alpha e^{-2K_\alpha (\zeta - \zeta_f)}. \tag{40}
\]

Despite the solution can be justified only at "small times," expression (35) provides a valuable information about the scaling, as it shows that the characteristic speed of the front is

\[
u_f = \frac{(1 + |K_\beta|^2) K_\alpha}{3k_\beta m |\sin \zeta_f + iK_\beta \cos \zeta_f|^2}. \tag{41}
\]

This is consistent with Eq. (27) at \( \zeta = 0 \). The dependence of \( \nu_f \) on \( m \) and \( \chi \) is shown in Fig. 4. It is interesting that for subsonic waves, \( \delta < a_\gamma \) (\( \chi < \pi/2 \), these dependences are

\[
\text{FIG. 4. The dependences of the dimensionless initial front velocity on parameter } m \text{ at different } \chi \text{ shown near the curves for the AIT regime at } h = \infty, k_\beta = 1/3.
\]
monotonic, while for super-resonant bubbles, \( a_0 > a_r (\chi > \pi/2) \), there exists a maximum near \( m = 1 \). The peak value increases as \( \chi \) approaches to \( \pi \) (the non-dissipative case), but the medium is effectively dissipative, as for super-resonant bubbles \( \text{Re}\{k^2\} < 0 \). For sub-resonant bubbles as \( \chi \to 0 \) we have \( K_0 \to 0 \) and the front velocity tends to zero. It also becomes zero for the case of super-resonant bubbles at \( \chi = \pi \) and \( m < 1 \). The critical value \( m = 1 \) at \( \chi = \pi \) corresponds to transition from an acoustically transparent to acoustically non-transparent medium. Asymptotics of the front velocity at small and large \( m \) follow from Eqs. (41) and (22),

\[
u_f \sim \frac{1}{3k_\mu} \begin{cases} \sin \chi, & m \ll 1, \\ m^{-1/2} \sin \frac{\chi}{2}, & m \gg 1. \end{cases}
\] (42)

### IV. NUMERICAL STUDY

#### A. Method

To solve Eq. (18), we developed and tested a Lagrangian-Eulerian method. From the point of view of mechanics of multiphase systems an \( M \)-fractional system can be treated as a \((M+1)\)-phase continuum (e.g., Ref. 4). The kinematic and continuity equations for each fraction can be written in the form

\[
\frac{d\tilde{\xi}_j}{d\tau} = u_j, \quad \frac{d\beta_j}{d\tau} = -\beta_j \frac{\partial u_j}{\partial \tilde{\xi}_j}, \quad \frac{d\tau}{d\xi} = \frac{\partial u_j}{\partial \xi} + u_j \frac{\partial}{\partial \xi},
\]

\[j = 1, \ldots, M.\]
(43)

Here each fraction of bubbles is a continuum with Lagrangian coordinate \( \tilde{\xi}_j \) and Eulerian coordinate \( \xi_j = \xi_j(\tilde{\xi}_j, \tau), \) \( j = 1, \ldots, M \). The Lagrangian coordinate for each fraction can be selected to be equal to the Eulerian coordinate at \( \tau = 0 \), so all fractions can be indexed with the same Lagrangian coordinate \( \tilde{\xi}_j = \xi_j = \xi_j(\xi, \tau), \xi_0(\xi, 0) = \xi = \xi \). Also we have \( \beta_j(\xi, 0) = \beta_j^0 \), so we have all initial conditions.

The right-hand sides of Eq. (43) require computation of functions \( u_j(\xi, \tau) \) and their derivatives. We have from Eq. (18),

\[u_j = \frac{1}{3k_\mu(\text{Re}b_j)} f_j, \quad f_j = \text{Re} \left\{ e^{iK_0} A \frac{\partial A}{\partial \xi} \right\}_{\xi = \xi_j} + g_j,\]
(44)

Multiplying this expression by \( k_{Rj} \) defined in Eq. (17), we obtain

\[\text{Re} b_j = \frac{1}{3k_\mu(\text{Re}b_j)} q_j, \quad q_j = k_{Rj} |f_j|,\]
(45)

Function \( q(R) = 3k_\mu(R)R \) can be tabulated and used for interpolation of the inverse function \( R(q) \) for arbitrary input. So, given \( q = q_j \) one can determine \( \text{Re} b_j \) and then \( u_j \),

\[u_j = \text{sign}(f_j) \frac{1}{k_{Rj}} R(q_j),\]
(46)

where \( \text{sign}(f) \) denotes the sign of \( f \). We have then

\[
\frac{\partial u_j}{\partial \xi} = \frac{\text{sign}(f_j)}{k_{Rj}} R'(q_j) \frac{\partial}{\partial \xi} \left( k_{Rj} |f_j| \right) = R'(q_j) \frac{\partial f_j}{\partial \xi},
\]

\[
= R'(q_j)c_j \text{Re} \left\{ e^{iK_0} \left( \frac{\partial A}{\partial \xi} \right)^2 + A \left( \frac{\partial \tilde{A}}{\partial \xi} \right)^2 \right\}_{\xi = \xi_j},
\]

\[
= \frac{c_j}{q'(\text{Re}b_j)} \text{Re} \left\{ e^{iK_0} \left( \frac{\partial A}{\partial \xi} \right)^2 - \tilde{K}^2 |A|^2 \right\}_{\xi = \xi_j}.
\] (47)

For approximation (5), functions \( q(R) \) and \( q'(R) \) in Eqs. (46) and (47) are

\[q(R) = 1 + 0.9R^{1/2}/1 + 0.3R^{1/2}, \quad q'(R) = 1 + 1.5R^{1/2} + 0.27R/(1 + 0.3R^{1/2})^2.
\] (48)

To compute \( A \) and \( \partial A/\partial \xi \) we used a solver for the Helmholtz equation on a non-uniform grid with a piecewise constant wave number. The idea of the solver is the following. Assume that the computational domain \([\xi_0, h]\) is subdivided into \( L \) pieces by the grid points \( \xi_j = \xi_j^{(0)} < \xi_j^{(1)} < \cdots < \xi_j^{(L)} = h \). Assume then that the partitioning is performed in a way that all jumps in the wave number \( K \) correspond to some grid points and the grid is dense enough to tolerate piecewise constant approximation of \( K = K(\xi), \xi \in [\xi_j^{(l-1)}, \xi_j^{(l)}], l = 1, \ldots, L \). We have then for each piece

\[A(\xi) = A^{(l)}(\xi) = C_1^{(l)} e^{iK_0^{(l)}}, \quad C_2^{(l)} e^{-iK_0^{(l)}}, \xi \in [\xi_j^{(l-1)}, \xi_j^{(l)}].\]
(49)

These solutions can be matched by requesting continuity of \( A \) and \( \partial A/\partial \xi \) at all intermediate grid points, while satisfying the boundary conditions at \( \xi = \xi_f \) and \( \xi = h \), i.e.,

\[A^{(l)}|_{\xi = \xi_j^{(l)}} \sin e^{\theta_j} + \frac{\partial A^{(l)}}{\partial \xi}_{\xi = \xi_j^{(l)}} \cos e^{\theta_j} = 1,\]

\[A^{(L)}|_{\xi = \xi_j^{(L)}} = 0, \quad A^{(l)}|_{\xi = \xi_j^{(l)}} = A^{(l+1)}|_{\xi = \xi_j^{(l)}}, \quad l = 1, \ldots, L - 1.\]
(50)

The boundary condition at \( \xi = h \) is as stated in Eq. (20), while at \( \xi = \xi_f \) we used the Robin boundary condition, which follows from the matching of solutions for bubbly and pure liquids [see Eq. (32)]. This results in a sparse linear system of \( 2L \) equations for \( 2L \) unknowns \( C_1^{(l)} \) and \( C_2^{(l)} \) (the four-diagonal system can be reduced to a three-diagonal system). According to (49) the solution of the system provides an accurate evaluation of \( A(\xi) \) and \( \partial A/\partial \xi \) at any intermediate point \( \xi \).

Now let us summarize the numerical procedure. At \( \tau = 0 \), we introduce a uniform grid of \( N + 1 \) points \( \xi_n = nh, h = h/N, n = 0, \ldots, N \). This grid initializes \( (N + 1)M \) variables \( \xi_j^{(n)} = \xi_j^{(n)}(\xi) \). System (43) then turns into the following system of the ODE’s

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\[ \frac{d\zeta_j^{(n)}}{d\tau} = u_j^{(n)}, \quad \frac{d\beta_j^{(n)}}{d\tau} = -\beta_j^{(n)} \left( \frac{\partial u_j}{\partial \zeta_j} \right)^{(n)}, \]

\[ \zeta_j^{(n)}\big|_{\tau=0} = z_j^{(n)}, \quad \beta_j^{(n)}\big|_{\tau=0} = \beta_{j0}, \quad n = 0, \ldots, N, \quad j = 1, \ldots, M. \] (51)

For explicit solvers, such as the Dormand-Prince method (MATLAB ode45 procedure) used in the present study, given variables \( f_j^{(n)} \) and \( b_j^{(n)} \) the right-hand side should be computed. This computation requires summation of the distributions \( b_j \) to obtain \( K^2 \) in Eq. (18). However, in the Lagrangian approach the values of \( b_j \) are available only at \( z_j = z_j^{(n)} \), which for different fractions \( j \) are different. Therefore, some procedure is needed to obtain \( b_j \) sampled on a common for all fractions grid. We successfully used the following simple method.

First, we created a mesh for the mixture, which is a union of the meshes for all fractions (it contains \((N + 1)M\) points if there are no duplicates). Then for each fraction \( j \) a subset of the mixture mesh with points \( z_j \in [\min\{z_j^{(n)}\}, \max\{z_j^{(n)}\}] \) was identified, and \( b_j \) was evaluated at \( z_j^{(n)} \) using samples \( b_j^{(n)} = b_j(z_j^{(n)}) \) (we used a standard cubic interpolation). Contributions from all fractions were summed up with weights \( m_j e^{\gamma_j} \) to obtain \( K^2 \). In the Helmholtz equation solver described above an automated procedure for grid coarsening (consistent with quasi-constant approximation of the wave number) was applied. We also defined \( \zeta_f \) for the mixture as \( \zeta_f = \min_j \{z_j^{(n)}\} \) (the left end of the computational domain).

### B. Monodisperse systems

We studied a number of cases with different sets of controlling parameters \((m, \chi, h)\) and found that Fig. 2 describes well enough the regimes observed. In all computations the dimensionless parameter \( g \), responsible for the gravity was small \((g = 0.005)\). Typical examples of computations are provided in Figs. 5 and 6. The former figure shows the clustering mode for sub-resonant \((\chi < \pi/2)\) and super-resonant \((\chi > \pi/2)\) bubbles. The latter figure displays the AIT mode. The case illustrated in the upper plot of Fig. 6 shows that while the AIT shock wave forms and propagates, bubbles cluster near the pressure antinode closest to \( z = h \). At higher values of the attenuation coefficient, this effect may not be present or shows up at larger times when the length of the region occupied by bubbles becomes substantially short. These figures also show that different regimes can be
realized only by varying parameter \( \chi \), which does not depend on the void fraction or acoustic amplitude (it heavily depends on the bubble size). It is also remarkable that computations using the present model are limited in time by the reason that the void fraction forms singularities either at the front of the AIT wave or somewhere else due to clustering as seen in Figs. 5 and 6. To treat such singularities and extend the computational time one should extend the model by describing the behavior of the layers with a high volume fraction of bubbles, which can be considered as a sort of foam. Also, bubble collisions and possible coalescence and fragmentation in these regions may play a role. Such modeling goes far beyond the model of the present paper, and we simply state that computations stop at a certain point, when the void fraction becomes larger than it is allowed by the model.

Figure 7 illustrates the effect of a boundary condition at \( \xi = h \) on the volume fraction of the bubbles and the sound amplitude.

Note also that the front velocity and position for the case computed illustrated in Fig. 7 are in excellent agreement with the asymptotic solution (36) and even at \( s = 25 \) the relative difference is only 2%, which can be expected at low \( m \).

We checked the constraint (41) for the analytical solution and we have found that \( 2K_0q_0f \approx 1 \) at \( \tau = 25 \), so the asymptotic result can be used for estimations not only at \( 2K_0q_0f < 1 \).

C. Polydisperse systems

While in many cases polydisperse systems can be modeled within the framework of monodisperse mixture,
generally speaking, their acoustic properties can differ drastically (e.g., Ref. 3). In any case, as soon as the mixture is described as a medium with dispersion and dissipation, the pressure and bubble velocity fields at \( t = 0 \) can be found from Eqs. (22)–(25) and further analysis of the small times and possible regimes can be performed. In these equations and analysis, one should use effective parameters \( m \) and \( \gamma \) based on the complex wave number for the polydisperse mixture, so

\[
m_{\text{eff}} e^{i \lambda_{\text{eff}}} = k_0^2 - 1 = \sum_{j=1}^{M} \beta_j m_j e^{i \lambda_j}.
\]  

(53)

The problem then occurs with the velocity, as each fraction of bubbles has a substantially different velocity characterized by parameters \( c_j \) and \( k_j (\text{Re} \eta_j) \) [see Eq. (18)]. So, in an acoustic field, which is affected jointly by all fractions, each fraction moves with its own velocity. Certainly, this should lead to the dispersion of the AIT wave velocity, and possibly to several jumps moving at different speed. Motion of fractions with different velocities leads to dynamics of the bubble size distribution, which can create a very different picture from what we observed for monodisperse systems. Because of substantially large number of parameters characterizing polydisperse systems, for illustrations described below we used parameters of air-water systems at atmospheric pressure and room temperature \( (p_a = 100 \text{kPa}, T_a = 293 \text{K}) \) for acoustic fields of frequency \( f = 200 \text{kHz} \) and amplitude \( \epsilon = 0.2 \). We varied only the initial bubble volume content and the bubble size distribution.

Figure 9 shows the clustering and the AIT modes for two fractional mixtures. The two qualitatively different cases shown correspond to the mixtures with the same parameters except for the bubble sizes, which are closer to resonance in the second case and so the attenuation in the second case is higher. For both cases \( m_{\text{eff}} \) and \( \lambda_{\text{eff}} \) calculated according to (53) indicate the AIT mode for the given \( h = 20 \) \((m_{\text{eff}} = 0.09, \lambda_{\text{eff}} = 0.8 \) and \( m_{\text{eff}} = 0.56, \lambda_{\text{eff}} = 0.83 \), respectively). However, such a mode appears to be unstable for the first case, and bubbles of different sizes cluster near the nodes and antinodes of the acoustic wave (for this illustration we selected a mixture of sub-resonant and super-resonant bubbles). The graph at the bottom of Fig. 9 illustrates the AIT mode for the two fractional mixture. It is clearly seen that there are two fronts moving with different velocities. It is interesting to note that behind the front of the faster wave the mixture becomes free from the bubbles of a certain size. This results in a substantial change of the acoustic properties of the medium (it becomes acoustically transparent), which in its turn causes the regime change from the AIT observed initially to the clustering at the later times (e.g., we can observe oscillations of the void fraction between the two wave fronts; at certain parameters the direction of propagation of the slowest wave also can change).

This effect is also well seen in Fig. 10 displaying the computational results for a four-fractional mixture \((m_{\text{eff}} = 0.24, \lambda_{\text{eff}} = 0.86, h = 25) \). Here, the acoustic field “cleans” the mixture from the bubbles of nearly resonance sizes (another interesting effect). Behind the front of the fast wave driving such bubbles out of the mixture we have a bubbly medium, which is not optically transparent (the bubbles of sizes substantially smaller than the resonance size remain in the liquid), but the mixture becomes “acoustically transparent,” since it does not contain the nearly resonance bubbles heavily contributing to the dissipation. The graph at the bottom of Fig. 10 shows that the attenuation of the acoustic wave amplitude does not change significantly behind the fronts of the fast waves, while the attenuation in the liquid containing the nearly resonance bubbles is substantial.

The real mixtures normally have much more bubbles of different sizes, and continual bubble size distributions can better describe situations encountered in practice. Such distributions can be considered as limiting cases when the number of discrete fractions tends to infinity. For the numerical

![FIG. 9. The clustering and the AIT regimes in two-fractional polydisperse mixtures (air-water mixtures at \( p_a = 100 \text{kPa}, T_a = 293 \text{K} \) in an acoustic field \( f = 200 \text{kHz}, \epsilon = 0.2 \), \( \lambda_{\text{eff}} = 10^{-4}, h = 20 \), \( \nu_j \) is the relative number of bubbles of \( j \)-th fraction in the mixture, \( \nu_j = n_{j,0}/n_{\text{tot}} \), \( j = 1, 2 \).](image)

![FIG. 10. A mixed regime in a four-fractional polydisperse mixture (in the same conditions as in Fig. 9), with fractional content \( \nu_1 = 0.9, \nu_2 = 0.03, \nu_3 = 0.02, \) and \( \nu_4 = 0.05 \), \( \lambda_{\text{eff}} = 10^{-4}, h = 25 \). The plot on the top shows volume fraction distributions. The total bubble volume fraction and the sound amplitude are shown on the bottom plot.](image)
modeling of this case shown in Fig. 11, we used \( M = 101 \) fractions sampling the standard normal distribution over sizes in the interval \([a_m - 3\sigma, a_m + 3\sigma]\) and uniform in space at \( \tau = 0 \),

\[
\begin{align*}
n_{j00} &= C \exp \left( -\frac{(a_0 - a_m)^2}{2\sigma^2} \right), \\
a_{j0} &= a_m - 3\sigma \left( 1 - \frac{j - 1}{M - 1} \right), \quad j = 1, \ldots, M,
\end{align*}
\]  

(54)

where \( a_m \) and \( \sigma \) are the mean and standard deviation, and \( C \) is the constant depending on \( \sigma \), \( a_m \), and \( M \), which can be determined from the specified initial bubble volume fraction \( x_{00} \) [see Eq. (10)]. Figure 11 illustrates the effect of parameter \( \sigma \) at fixed \( a_m \) on the propagation of the AIT wave. At \( \sigma \to 0 \), we have the case of monodisperse liquid, where a sharp peak of the void fraction forms and propagates indicating the position of the wave front. As \( \sigma \) increases the dispersion of the bubble velocity increases, and instead of a shock wave we have some continuous profile of the wave of void fraction. It is noticeable that the velocity of the peak increases, while its magnitude decreases at the increasing \( \sigma \). In fact, the continuous profile of the wave consists of many tiny shocks related to each bubble fraction, which is seen as some small oscillations on the computed results for larger \( \sigma \). These oscillations can be smoothed out by using a greater number of samples \( M \). Our computation show that at relatively large \( x_{00} \) due to a strong dissipation the front velocity becomes very slow and all moving bubbles are located in some vicinity of the front, which makes the effect of bubble velocity dispersion less significant and computations for polydisperse mixtures with some effective bubble radius agree well enough with those for monodisperse systems.

Finally, we note that the conditions of the experiments\(^{27}\) where the AIT mode was observed are out of the range of the present theory, which is valid for small enough sound amplitudes. Nonetheless, estimations of the parameters realized in experiments (using a 12-fractional histogram for the bubble size distribution with the volume mean bubble radius \( a_m = 18.5 \mu m \) and the overall initial volume fraction \( x_{00} = 4 \times 10^{-3} \)), the height of the experimental setup (\( H = 0.027 m \)), and the range of the acoustic frequencies show that the effective mixture parameters are located deeply inside the region for the AIT regime shown in Figs. 2 and 3. For example, at frequencies \( f = 209.2 \) kHz and \( 89 \) kHz realized in experiments we have \( m_{eff} = 69, \ h_{eff} = 2.2, \ h = 24 \) and \( m_{eff} = 150, \ h_{eff} = 0.57, \ h = 10 \), respectively. The experimental observations qualitatively agree with the present theory, where the front of the AIT wave was clearly indicated by a thin region of high bubble concentration. We also remark that despite a satisfactory agreement with experiments, computations using discrete weakly nonlinear collisionless models of bubbly liquids\(^{26,27}\) should be studied more systematically both from the theoretical and computational point of view.

V. CONCLUSION

The present analytical and numerical study identifies two basic regimes of bubble self-organization in acoustic fields, which we termed as the clustering and the AIT modes. We found analytically the boundaries between these two modes in the space of parameters for monodisperse systems and checked numerically that such regimes exist. The numerical study showed that some combinations of the two modes may also occur. This is particularly the case for polydisperse mixtures consisting of two and more fractions of bubbles, which parametric study is complicated by the large dimensionality of the parameter space. Because of a strong dispersion of the bubble velocity in polydisperse mixtures, the structure of the volume concentration waves may have several shocks or do not have shocks at all (for continuous distributions). For the numerical analysis, an algorithm, which enables accurate solution of a system of nonlinear wave equations, was developed and tested. The model and, respectively, the algorithm have limitations, as they do not prevent formation of singularities of void fraction. Modifications of the model can be done by detailed consideration of physics of highly concentrated bubbly layers, which goes beyond the present study and can be considered as a future task.
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