FMM accelerated BEM for 3D Helmholtz equation

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Problems resulting in the Helmholtz equation

While here we are interested with acoustics, many parabolic and hyperbolic problems in frequency domain (time harmonic solutions) result in the Helmholtz equation

\[ \nabla^2 \phi + k^2 \phi = 0, \quad x \in V \subset \mathbb{R}^3, \quad k \in \mathbb{R}, \]

**Sommerfeld radiation condition (infinite domains)**

\[ \lim_{|x| \to \infty} \left( |x| \left( \frac{\partial \phi}{\partial |x|} - ik \phi \right) \right) = 0. \]

Conventional acoustics (wave equation):

\[ \Delta p(x, t) = \text{Re}\{e^{-i\omega t}\phi(x)\}, \quad k = \frac{\omega}{C}. \]

Relaxating media (complex fluids)

Heat/mass diffusion

Electrodynamics (Maxwell equations)

Pair of the Helmholtz equations for the Debye potentials

Quantum mechanics (Shroedinger & Klein-Gordon equations)

Real or purely imaginary wavenumber (Yukawa potential)
Boundary Integral Equations
(Closed surfaces, for direct BEM)

Green's identity:
\[ \pm \phi(y) = L[q] - M[\phi], \quad y \notin S, \]
\[ \pm \frac{1}{2} \phi(y) = L[q] - M[\phi], \quad y \in S, \]

Single layer potential:
\[ L[q] = \int_S q(x)G(x, y)dS(x), \]

Double layer potential:
\[ M[\phi] = \int_S \phi(x) \frac{\partial G(x, y)}{\partial n(x)}dS(x), \]

Green's function:
\[ G(x, y) = \frac{e^{ikr}}{4\pi r}, \quad r = |x - y|. \]

Combined (Burton-Miller) BIE:
\[ \pm \frac{1}{2} [\phi(y) + \lambda q(y)] = (L + \lambda L')[q] - (M + \lambda M')[\phi], \]

Derivatives of single and double layer potentials:
\[ L'[q] = \int_S q(x) \frac{\partial G(x, y)}{\partial n(y)}dS(x), \quad M'[\phi] = \frac{\partial}{\partial n(y)} \int_S \phi(x) \frac{\partial G(x, y)}{\partial n(x)}dS(x). \]

Generic boundary conditions:
\[ \alpha(x)\phi(x) + \beta(x)q(x) = \gamma(x). \]

\((\alpha, \beta, \gamma \text{ are given})\)
\[ \beta = 0: \text{Dirichlet} \]
\[ \alpha = 0: \text{Neumann} \]
\[ \alpha, \beta = \text{const}: \text{Robin} \]
Boundary Integral Equations
(Arbitrary surfaces, for indirect BEM)

Solution as a sum of single and double layer potentials:

\[ \phi(y) = L[\sigma] + M[\mu], \quad y \notin S, \]

BIE (jump conditions):

\[ \phi^\pm(y) = L[\sigma](y) + M[\mu](y) \pm \frac{1}{2} \mu(y), \quad y \in S, \]
\[ q^\pm(y) = \mp L'[\sigma](y) \mp M'[\mu](y) + \frac{1}{2} \sigma(y), \quad y \in S. \]

+ Generic boundary conditions on each side
Discretization

- Sampling by several elements per wavelength necessary for accuracy (say 5-10);
- So, the number of elements, $N_{el} = O((kD)^2)$;
- This grows proportional to the square of the acoustic frequency;
- And to the square of the domain size;
- E.g. Human head + torso acoustic scattering computation at 20 kHz requires meshes with $N_{el} \sim 500,000 - 1,000,000$. 

\[
\delta \sim \sqrt{\frac{4\pi a^2}{N_{el}}}
\]
\[
k\delta \ll 1, \quad k = \frac{2\pi}{\lambda} = \frac{2\pi f}{c}
\]
\[
N_{el} \gg 4\pi (ka)^2
\]
\[
f \sim 20 \text{ kHz}, \quad a \sim 0.1 \text{ m}
\]
\[
ka \sim 30, \quad N_{el} \gg 10^4
\]

For complex shapes (large surface area) the number of boundary elements can be much larger.
### Standard and Fast Multipole accelerated BEM

Before going to details of the FM BEM, look at the table

<table>
<thead>
<tr>
<th>Task</th>
<th>Standard BEM</th>
<th>FM BEM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reformulate the problem in terms of BIE</td>
<td>✔️</td>
<td>✔️</td>
</tr>
<tr>
<td>Discretize the boundary</td>
<td>✔️</td>
<td>✔️</td>
</tr>
<tr>
<td>Compute and store boundary integrals</td>
<td>Full storage, memory (\sim (kD)^4)</td>
<td>Partial storage, memory (\sim (kD)^2)</td>
</tr>
<tr>
<td>Solve linear system</td>
<td>If direct (\sim (kD)^6), iterative (\sim N_{\text{iter}} (kD)^4)</td>
<td>Iterative (\sim N_{\text{iter}} (kD)^2), efficient FMM preconditioner</td>
</tr>
</tbody>
</table>

Max solvable problem size (PC):  
- Standard BEM: \(N \sim 3 \cdot 10^4 (kD\sim 10^2)\)  
- FM BEM: \(N \sim 3 \cdot 10^6 (kD\sim 10^3)\)
What the FMM does?

- Computes $N \times N$ matrix-vector product, $Ax$, for cost less than $O(N^2)$ (ideally $O(N)$ or $O(N \log N)$);

- The catch is in the controlled accuracy (which can be machine precision, or lower resulting in substantial speedups);

- The matrix is decomposed into sparse and dense parts, $A = A_{\text{sparse}} + A_{\text{dense}}$;

- The sparse matrix represents interaction of closely located elements (some neighborhoods); $A_{\text{sparse}}x$ can be computed in $O(N)$ operations and may require $O(N)$ memory;

- The dense matrix represents interaction of far elements (outside the neighborhoods); $A_{\text{dense}}x$ can be computed in $O(N \log^a N)$ operations and requires $O(\log N)$ memory if done efficiently;

Basics of the FMM and specifics for the Helmholtz equation can be found in our book.
Peculiarities of our direct FMBEM

Gumerov & Duraiswami, JASA, 2009

• BEM itself is an approximate method, due to
  • geometric discretization errors (e.g. flat boundary elements);
  • approximate computation of boundary integrals (use of collocation methods);
  • iteration tolerance (large systems cannot be solved directly);

• The FMM can be tuned to provide a consistent accuracy;

• Non-singular integrals can be computed using a low order quadrature (center point or trapezoidal quadratures);

• Diagonal elements of singular operators can be computed for the cost of 4 FMM runs, using a novel method of test functions described in the paper;

• Low accuracy/low cost FMM can be used for preconditioning in the inner loop of flexible GMRES iterative method;

• A new version of the FMM scaled close to $O((kD)^2)$ for $kD$ up to $10^3$ is developed and tested.
Performance of the direct FMBEM
Gumerov & Duraiswami, JASA, 2009

$kD = 0.1 – 500$, plane wave scattering off a sphere

Practical FMM is scaled close to $O((kD)^2)$. The BEM is scaled as $O((kD)^{2.4})$ due to the increasing number of iterations at the increasing $kD$. 

$240,002$ vertices
$480,000$ elements

$kD = 104$
(ka = 30)

Comparison with analytical solution
Scaling of the FMM with frequency

• There exist some versions of the FMM for the Helmholtz equation which are scaled as $O(N)$, but applicable only for low frequencies, so they are scaled poorly with respect to $kD$.

• To achieve scaling close to $O((kD)^2)$ only $O(p^2)$-type translation methods can be used in the FMM (based on the diagonal forms of the translation operators).

• Such algorithms use high/low frequency switches for wideband computations.

($p$ is the series truncation number in the FMM, so the contribution of elements in the far field is represented via $p^2$ terms)

High/low frequency switches used in Gumerov & Duraiswami, JASA, 2009.

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Computation of the head related transfer function (HRTF) for audible frequency range

Gumerov, O’Donovan, Duraiswami, Zotkin, JASA, 2010

(The total run time through the entire frequency range is 70 hours on a 4 core PC with 8GB RAM)

Comparison of computations and experiments (Azimuth = 0°)

Mesh: 445,276 elements

Mesh: 117,596 elements

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Indirect FMBEM for Helmholtz Equation

• All kind of boundary integrals (regular, nearly singular, weakly singular, singular, and hypersingular) for the neighborhoods are computed analytically;

• Larger storage is needed for accurate BEM computations, but this is still $O(N)$.

• Details will be presented in upcoming paper of Gumerov, Adelman, Duraiswami, ICA/POMA 2013 (accepted)/ and extended version to be submitted to JASA 2013
Computation of acoustic bidirectional reflectance distribution function (BRDF) for wavy surfaces via Indirect FMBEM

Gumerov, Duraiswami, ASA Meeting 159, 2010

Sinusoidal surface
(Case of Sakuma, et Al, 2009)
diameter = 2.655 m,
amplitude= 0.0256 m.
period = 0.177 m

Incident wave:
\[ f = 2 \text{ kHz}, \]
\[ \theta' = 31.5^\circ, \]
\[ \phi = 181.5^\circ. \]
Indirect FMBEM simulations of scattering from a parabolic antenna and a sphere
Conclusion

• Developed and tested a fast multipole accelerated boundary element method for the Helmholtz equation in 3D, both in direct and indirect formulations
• The FMM acceleration and memory reduction enables BEM solution with millions elements on a contemporary multicore PCs
• Scaling of the FMBEM codes with frequency (theoretically as $O((kD)^2)$) is very important and can be achieved only via careful implementation of efficient translation schemes
• More work is needed for efficient FMBEM including hardware acceleration (e.g. graphics processors) and algorithms … this is the subject of our ongoing work